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Parabola-Based Discrete Curvature Estimation †

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Abstract

The local geometric properties such as curvatures and normal vectors play important roles in analyzing the local shape of objects. The result of the geometric operations such as mesh simplification and mesh smoothing is dependent on how to compute the curvatures of meshes, because there is no exact definition of the discrete curvature in meshes. In this paper, we indicate the fatal error in computing sectional curvatures of the most previous discrete curvature estimations. Moreover, we present a new discrete sectional-curvature estimation to overcome the error, which is based on the parabola interpolation and the geometric properties of Bezier curve.

***Keywords**: Discrete Curvature and Parabolic Interpolation*

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1. Introduction

The problem of estimating the geometric properties such as normal vectors and curvatures in triangular meshes plays important role in many applications such as surface segmentation, anisotropic remeshing or non-photorealistic rendering. A lot of efforts have been devoted to this problem, but there is no consensus on the most appropriate way[1,3,4,5,8,9]. Popular methods include quadratic fitting, where the estimated curvature is the one of the quadratic that best fits a certain neighborhood of a vertex locally. Most recently, Goldfeather propose the use of a cubic approximation scheme which takes into account vertex normals in the 1-ring. The accuracy of these curvature estimations is dependent of that of fitting. If the one-ring neighborhood has many vertices or has a oscillated shape, then the approximated surface does not resemble the local shape and these estimations may yield a high error. Other methods typically consider some definition of curvature that can be extended to the polyhedral setting. These methods compute Gaussian curvature and Mean curvature based on the Gauss-Bonnet theory and Euler theory. Taubin presented a method to estimate the tensor of curvature of a surface at vertices of a mesh[6]. Watanabe proposed a simple method of estimating the principal curvatures of a discrete surface[7]. Meyer et. al proposed a discrete analog of the Laplace-Beltrami operator to estimate the discrete curvature[2]. Most of these methods compute directly the sectional curvatures for each adjacent edge of a vertex. They assume the normal curve interpolate both the given vertex and an adjacent vertex and the curve is represented by Taylor series. However, they make the same mistake that they adopt the distance between the given vertex and its adjacent neighbor vertex as the parameter of the series. Figure 1 shows their drawback. There are several polygons of different interior angles, all of which are circumscribed by circles of the same radius. The discrete curvatures estimated by those methods are the same as that of the circle. It is quite alien to universal concepts.

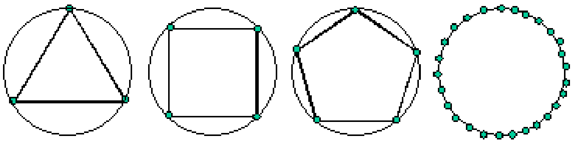

Figure 1: *Several Polygons with the same discrete curvature*

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In this paper, we present symmetric parabola-based discrete curvature estimation in order to solve such a problem in computing the sectional curvatures by the previous discrete curvature estimations. Our method is based on the parabola interpolation. We show that our method has a good geometric property so that we may derive a more simple formula and resembles the local shape better than the previous methods. Moreover, we detect a fatal error of the circle-based discrete curvature estimations.

This paper is organized as follows: we indicate an error of the circle-based discrete curvature estimations in Section 2. In Section 3, we present a new discrete curvature estimation that is based on the parabola interpolation and has a good geometric property. Moreover, the comparison of our method and circle-based estimation is shown in Section 4. Finally, we conclude the paper with some remarks in Section 5.

2. Review of Circle-Based Discrete Curvature

We review the circle-based discrete curvature estimation adopted in the previous methods of estimating the discrete curvature of meshes. This method utilizes the Taylor series of a curve and adopts the distance between two vertices as parameter of the curve. It makes the trajectory of points with the same curvature be a circle. So, in this paper we call this method as C-discrete curvature.

2.1. C-Discrete Curvature Formula

Most of the previous discrete curvature estimation compute directly the sectional discrete curvature whether they use the tensor of curvature or Laplacian Operator [2,3,6]. All of them compute the curvature by using the Taylor series of a curve interpolating two vertices. Let p and p_i be a given vertex and its adjacent vertex, respectively and let $g(s)$ be a continuous curve passing through the two vertices: $g(0) = p$, $g(s) = p_i$. Then the curve may be represented by its Taylor series:

$$\begin{aligned} g(s) &= g(0) + sg'(0) + \frac{s^2}{2}g''(0) + O(s^3) \\ &= g(0) + sT + \frac{s^2}{2}\kappa(p)N + O(s^3), \end{aligned}$$

where T and N are the unit tangent vector of the unit normal vector, respectively, and $\kappa(p)$ is the sectional curvature of $g(s)$ at point p to the direction pp_i. By applying the inner product to both sides with N, we may get the following equation

$$N\cdot(g(s)-p) = sN\cdot T + \frac{s^2}{2}\kappa(p)N\cdot N + O(s^3).$$

Because $N \perp T = 0$ and $N \perp N = 1$, the above equation may be changed to the following simple equation

$$\begin{aligned} N\cdot(g(s)-p) &= \frac{s^2}{2}\kappa(p) + O(s^3) \\ \kappa(p) &= \frac{2N\cdot(g(s)-p)}{s^2} + O(s). \end{aligned}$$

The previous methods define the C-discrete curvature $\kappa_C(p)$ by assuming that the parameter of the series is the distance between two vertices:

$$\kappa_C^p(p_i) \equiv \frac{2N\cdot(p_i-p)}{||p_i-p||^2} \tag{1}$$

2.2. The Drawback of C-Discrete Curvature

First of all, we find out the set of points with the same discrete curvature α. For the convenience of computation, we assume that $p = (0,0)$, $p_i = (x,y)$ and $N = (0,-1)$. Then the formula of the C-discrete curvature of p is as follows:

$$\kappa_C(p) = \frac{2N\cdot(p_i-p)}{||p_i-p||^2} = \frac{2(0,-1)\cdot(x,y)}{||\sqrt{x^2+y^2}||^2} = \frac{-2y}{x^2+y^2} = \alpha.$$

Then,

$$x^2 + (y+\frac{1}{\alpha})^2 = \frac{1}{\alpha^2}$$

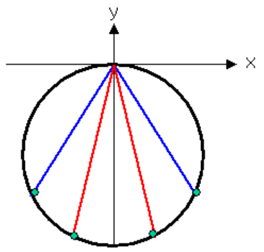

Figure 2: *The set of points with curvature* α

The trajectory is the circle of center $(0,-\frac{1}{\alpha})$ and radius $\frac{1}{\alpha}$ as shown in Figure 2. That is, whenever the adjacent vertex p_i is on the radius of the circle though it makes the different interior angle with the given vertex p, the sectional C-discrete curvature of p to the direction pp_i is the same as the that of the circle. In general, the interior angle of a triangle is sharper than those of square and octagon. So, we may consider that the curvature of a vertex in triangles is greater than that of other regular polygons(see Figure 1). The result of the C-discrete curvature estimation breaks the general concepts. Moreover, there is another drawback of this method that the range of the value is restricted. If $||pp_i|| \geq 1$, then $\frac{y}{x^2+y^2} \leq 1$. Hence,

$$\kappa_C^p(p_i) = \frac{2y}{x^2+y^2} \leq 2$$

Therefore, we have known that this method can not reflect the local shape although the adjacent vertices have a sharp interior angle because of the upper bound of curvature values.

Figure 3 shows an example which indicates the drawbacks of the C-discrete curvature estimation. The vertices of a polygon lie on the union of two circles of the same radius. By C-type estimation, there are two types of the curvature value: one is on the intersection of two circles, the other is on the other region. The former is a small value, which appears on the intersection of two circles, the later does on the other region. But, there are three types of interior angles. Though the most left vertex has a sharper interior angle than other vertices, it has the same discrete curvature as one of them.

- *It computes the same curvature whenever the vertices are on circles of the same radius.*
- *There is a upper bound of curvature values if the distance between two vertices is greater than 1.*

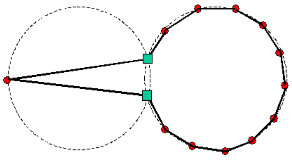

Figure 3: *counter-example of C-discrete curvature*

3. Parabola-Based Discrete Curvature

In this section, we introduce a new discrete curvature estimation based on the parabola interpolation. The formula has a good geometric perperty, so the curvature estimated by our method resembles well the local shape of polygons. In general, the local shape of a polygon at a vertex is determined by the geometric relationship between the vertex and its adjacent vertices. We have known that the concept of curvature is derived from a curve and it is a quantity to measure the local bending of curves. Therefore, the best method to resemble the local shape, following the original definition of a curvature is to use the quadratic curve interpolating the three consecutive vertices.

3.1. P-Discrete Curvature Formula

We adopt a quadratic Bezier curve as an interpolating curve because it has a good geometric property. Let A, B, C be three consecutive vertices. The general form of a quadratic Bezier curve is as follows:

$$P(t) = P_0 B_0^2(t) + P_1 B_1^2(t) + P_2 B_2^2(t),$$

where P_i are the control points of the Bezier curve and $B_i^n(t) = \frac{n!}{(n-i)!i!}(1-t)^{n-i} t^i$ are the Bernstein polynomials of degree n. In general, there are several methods to find the Bezier curve that interpolates the given vertices. That is, the conditions for two end vertices are already determined as follows: $P(0) = A, P(1) = C$, so the method is determined according to when the curve passes through the intermediate vertex B. This problem is a parameterization of curves. We can consider the folowing two methods:

standard : $P(\frac{1}{2}) = B$
length-based : $P(\frac{||AB||}{||AB||+||BC||}) = B$

The standard parameterization is simple to derive the good geometric relations, where the length-based parameterization resembles well the local shape but yields more complex formula. In this paper, we adopt the standard parameterization so that we may find out a good relationship between the curvature of a curve and the discrete curvature of a polygon.

First of all, we compute the three control points P_0, P_1, P_2 of the quadratic Bezier curve, satisfying the standard parameterization condition:

$$P_0 = A, \qquad P_1 = \frac{4B - A - C}{2}, \qquad P_2 = C.$$

Therefore, the interpolating Bezier curve is as follows:

$$P(t) = AB_0^2(t) + \frac{4B - A - C}{2} B_1^2(t) + CB_2^2(t).$$

The curvature of $P(t)$ at $t = \frac{1}{2}$ is

$$\begin{aligned} \kappa_P(\frac{1}{2}) &= \frac{||P''(\frac{1}{2}) \times P'(\frac{1}{2})||}{||P'(\frac{1}{2})||^3} \\ &= \frac{||4(A - 2B + C) \times (C - A)||}{||C - A||^3}. \end{aligned}$$

Hence, we can define a new Parabola-based discrete curvature of the given vertex B as follows:

$$\kappa_P(B) = \frac{||4(A - 2B + C) \times (C - A)||}{||C - A||^3}. \qquad (2)$$

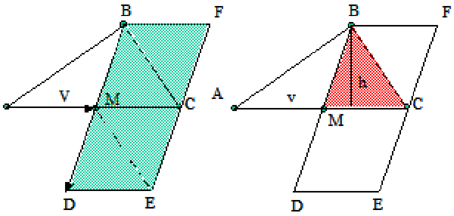

Figure 4: *The geometric meaning of P-discrete curvature*

3.2. Geometric Properties of P-Discrete Curvature

First of all, we find out the geometric properties of the P-discrete curvature formula. Let $V = \frac{C-A}{2}$ and $G = (A - 2B + C)$. The P-discrete curvature formula is

$$\kappa_P(B) = \frac{||4G \times 2V||}{||2V||^3} = \frac{||G \times V||}{||V||^3} = \frac{||G||||V||sin\theta}{||V||^3}, \qquad (3)$$

where θ is the in-between angle of the vectors G and V. The numerator of Eqn. (3) is the area of the parallelogram $BDEF$ and is four times as much as the area of the triangle BCF as shown in Figure 4. Therefore, the P-discrete curvature formula is

$$\kappa_P(B) = \frac{4\frac{1}{2}vh}{v^3} = \frac{2h}{v^2}, \tag{4}$$

where, h and v are the height and the width of the triangle BCF, respectively.

3.3. Validity of P-discrete curvature

In order to verify the propriety of the P-discrete curvature, we regularly sample n points on a circle of radius 1 and compute their P-discrete curvature. Let $p_i = (\cos\frac{2\pi i}{n}, \sin\frac{2\pi i}{n})$, $i = 0, \cdots, n-1$, be the vertices of a n-gon on the circle. By trigonometry, we can compute the values of v and h as follows:

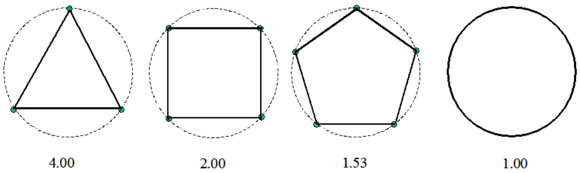

Figure 5: *Polygons with the different p-curvature values*

$$v = r\sin\frac{2\pi}{n}, h = r(1-\cos\frac{2\pi}{n}).$$

Therefore, as the number of sampling points increases to the infinity, the value of curvature at a vertex of the n-gon becomes that of the circle.

$$\lim_{n\to\infty}\kappa_{P(n)}(Q) = \lim_{n\to\infty}\frac{2(1-\cos\frac{2\pi}{n})}{\sin^2(\frac{2\pi}{n})} = \frac{1}{r}.$$

Figure **??** and Table 1 show the several polygons on a cicle of radius 1 and their P-discrete curvature values. This result is an excellent contrast to that of C-discrete curvature estimation (see Figure 1). The C-type estimation wishes that the curvature of the sampled vertices becomes that of a circle because the vertices are sampled on the circle. The method puts emphasis on the point of view that the vertices are on a circle. However, it goes against the concept of curvatures. It loses the information on the local shape. On the other hand, our method recognizes the vertices of polygons on the circle to have a sharper angle, not to be on a circle. That is, the P-discrete curvature of vertices of a triangle is 4 and that of retangle is 2.0. More the number of vertices increases to the infinity, less the curvature value decreases to 1. Therefore, our method resembles the local shape of vertices more than the C-discrete curvature estimation.

4. Symmetric Parabola-Based Discrete Curvature

There is an example that the P-discrete curvature estimation explained in Section 3 is not yet the perfect solution of resembleing the local shape of polygons. Figure 6 contains three polygons each of which has the same width $2v$ and the same height h. So, they have the same discrete curvature of the intermediate vertex B. However, one can expect that it has the largerst curvature among them because the last polygon has a sharper angle at the vertex than the others. In order to solve this imperfection, we propose the symmetric parabola-based discrete curvature estimation using a symmetric parabola.

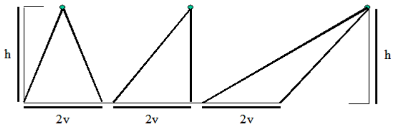

Figure 6: *Polygons with the same p-curvature values*

4.1. SP-Discrete Curvature Formula

Let N be the unit vector bisecting the interior angle of B

$$N = \frac{\frac{BA}{||BA||} + \frac{BC}{||BC||}}{||\frac{BA}{||BA||} + \frac{BC}{||BC||}||}.$$

Then, we can consider two right-angled triangles $\triangle AA'B$ and $\triangle CC'B$ as shown in Figure 7. One has the width of length v_A and the height of h_A, the other has the width of length v_C and the height of h_C:

$$\begin{aligned} h_A &= N \cdot BA, \qquad v_A = ||BA - (N \cdot BA)N|| \\ h_C &= N \cdot BC, \qquad v_C = ||BC - (N \cdot BC)N||. \end{aligned}$$

Then, we can consider the right-angled triangle $\triangle MM'B$ with the width $v_m = \frac{v_A+v_C}{2}$ and the height $h_m = \frac{h_A+h_C}{2}$. Therefore, we define the symmetric parabola-based discrete curvature of the vertex B as

$$\kappa_{SP}(B) \equiv \frac{2h_m}{v_m^2}. \tag{5}$$

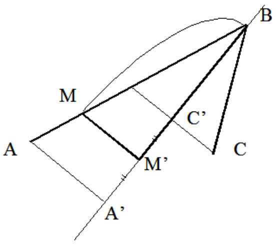

Figure 7: *Polygon and its Parabola*

4.2. Comparison of SP-DC and C-DC

In this subsection, we compare the symmetric parabola-based discrete curvature with the circle-based discrete curvature by using the Taylor series of curves interpolating the

n	3	4	5	6	7	8	9	10	11	100
curvature	4.00	2.00	1.5278	1.3333	1.2319	1.1715	1.1324	1.1055	1.0862	1.0009
Radius of curvature	0.25	0.50	0.6545	0.7500	0.8117	0.8535	0.8830	0.9045	0.9206	0.9990

Table 1: *The P-discrete curvature and its radius of curvature of the sampled points on the circle of radius 1*

given two points. The C-curvature adopts the distance BM between two points as the parameter of the curve, whereas the SP-curvature adopts the horizontal distance MM' between them (see Figure 7). First of all, we derive the analytic formula of the symmetric parabora-based discrete curvature by using the Taylor series. For the convenience of deriving, we assume that the bisecting unit vector is $N = (0,-1)$, a given vertex is $p = (0,h_m)$, and its adjacent vertex is $p_i(v_m,0)$. Then, the parabola $g(t) = (t, f(t))$ interpolating two vertices is symmetric so that the unit normal vector at p is the same as N and $f'(0) = 0$. Hence, the first and second derivatives of the parabola at the vertex p are $g'(0) = (1,0), g''(0) = (0, f''(0))$. Therefore, we can compute the curvature of the curve at the vertex p as follows:

$$\kappa_g(0) = \frac{||g''(0) \times g'(0)||}{||g'(0)||^3} = ||f''(0)||.$$

In order to compute the second derivative of $f(t)$, we use the taylor series of $f(t)$. Because $f(t)$is a quadratic function, the form is as follows:

$$\begin{aligned} f(t) &= f(0) + f'(0)t + \frac{f''(0)}{2}t^2, \\ f''(0) &= \frac{2(f(t)-f(0))}{t^2} = \frac{2h_m}{v_m^2}. \end{aligned}$$

Therefore, the symmetric parabora-based discrete curvature is the same as the curvature of the curve $g(t)$:

$$\kappa_g(0) = \frac{||g''(0) \times g'(0)||}{||g'(0)||^3} = ||f''(0)|| = \kappa_{SP}(p).$$

The differences of C-discrete curvature and SP-discrete curvature are shown in Table 2. It includes the formulae of curvatures, the parameters of the curves in the Taylor series, the trajectory of points with the constant curvature, the range of curvature values, and the comparison of their magnitudes.

5. Conclusion

The analysis on the local properties of $3D$ meshes plays an important role in the applications such as morphing, simplification, smoothing. In special, the curvature at a point on a surface may represent the shape of its neighborhood. However, there is an exact definition of the curvature at a vertex. So, one have to approximate the value as a discrete curvature. The common previous method compute directly the sectional curvatures for each one-ring neighbor, and then derive the gaussian curvature and the mean curvature using the sectional curvatures. All of them utilize the circle-based discrete curvature to compute the sectional curvature. In this paper, we find out a fatal mistake and propose the parabola-based discrete curvature estimation to resolve the problem. Our method may be the basis of normal vector estimation and segmentation of meshes.

References

1. Sylvain Petitjean, " A Survey of Methods for Recovering Quadrics in Triangle Meshes", *ACM Computing Surveys* , **34**(2),pp 211–262(2002).
2. Mark Meyer and Mathieu Desbrun and Peter Schroder and Alan H. Barr, "Discrete Differential Geometry Operators for Triangulated 2-Manifolds", *Proc of VisMath* (2002)
3. J. Goldfeather and V. Interrante, " A novel cubic-order algorithm for approximating principal direction vectors", *ACM Transaction on Graphics*, **23**(1), pp 45–63(2004)
4. Holger Theisel and Christian Rossl and Rhaleb Zayer and Hans-Peter Seidel, "Normal Based Estimation of the Curvature Tensor for Triangular Meshes", *Proc. of Pacific Graphics'04*, pp 288–297(2004)
5. Tatiana Surazhsky and Evgeny Magid and Octavian Soldea and Gershon Elber and Ehud Rivlin, " A Comparison of Gaussian and Mean Curvatures Estimation Methods on Triangular Meshes", *Proc. of 2003 IEEE International Conference on Robotics and Automation*, (2003)
6. Gabriel Taubin, "Estimating the Tensor of Curvature of a Surface from Polyhedral Approximation", *Proc. of the Fifth International Conference on Computer Vision*, (1995)
7. Kouki Watanabe and Alexander G. Belyaev, "Detection of Salient Curvatures Features on Polygonal Surfaces", *Computer Graphics Forum* , **20**(3), (2001).
8. David Cohen-Steiner and Jean-Marie Morvan, "Restricted Delaunay Triangulations and Normal Cycle", *ACM Symposium of Computational Geometry* , (2003).

	C-Discrete Curvature	SP-Discrete Curvature
Formula	$\kappa_C(B) = \frac{2N\cdot BA}{\lvert\lvert AB\rvert\rvert^2}$	$\kappa_{SP}(B) = \frac{2N\cdot BA}{\lvert\lvert BA-(N\cdot BA)N\rvert\rvert^2}$
Parameter	Distance	Horizontal Distance
Range	$\kappa_C(B) \leq 2$ if $\lvert\lvert AB\rvert\rvert \geq 1$	$\kappa_{SP}(B) < \infty$
Trajectory	circle	parabola
magnitude		$\kappa_C(B) \leq \kappa_{SP}(B)$

Table 2: *The comparison of SP-curvature with C-curvature*

9. M. Garland and P. S. Heckbert, ”Surface simplification using quadratic error metrics”, *SIGGRAPH 97*, pp 209–216(1997)