Points and vectors

Points may represent vertices of polyhedra, samples on a surface, or control points of a B-spline patch.

Vectors represent displacements between points. AB, the vector from point A to B, is. also written B–A. Furthermore BA=–AB.

Points, vectors, and operations that combine them are the common tools for solving many geometric problems. The fundamental operators on vectors are the **dot product**, the **cross product**, and the **mixed product**.

When formulating geometric solutions or algorithms, try using **point and vector expressions,** rather than their coordinates.

The **norm** of vector U will be written ||U|| and denotes its length. A vector is said to be **unit length** if its norm is 1. A vector with zero norm is said to be **null**. For example, AA is a null vector. We will use the notation |U|, to denote the unit vector U/||U||. The term **direction** usually refers to a unit length vector. Directions represent tangents to lines or curves, normals to planes or surfaces, base vectors of coordinate systems, and the columns of matrices that represent rotations and translations.

U•V is the **dot product** (also called the inner product) of the vectors U and V. U•V is a **scalar** equal to the product c |U| |V|, where c is the **cosine** of the angle between them. Therefore, the dot product of two unit vectors is the cosine of their angle. Thus, two non-null vectors are **orthogonal** if and only if their dot product is zero. Furthermore, the norm of a vector U is $(U•U)^{0.5}$. More generally, if U is a unit vector, then U•V measures the length of the **orthogonal projection** of V onto the direction U. To illustrate the power of the dot product, consider the problem of computing reflections. Given two unit vectors U and V, the vector S=2(U•V)V–U is the **symmetry** of U with respect to V. The construction is based on the observation that by definition, U+S is parallel to V and has a norm that is twice the length of the projection of U upon V. Therefore, when a light ray or particle moving in the direction U bounces off a surface with normal N, its new **reflected direction** will be U–2(U•N)N.

U□V is the **cross product** (also called the outer product) of U by V. When U and V are parallel, UxV is a null vector. Otherwise it is a **vector** orthogonal to both U and V. Its norm is the product s |U| |V|, where s is the **sine** of the angle between them. Note that if U is the upward vertical direction and V is the forward direction, then UxV, it points to the **left**. Two non-null vectors are **parallel** if their cross product is zero. To illustrate the power of the cross product, consider the problem of finding a vector N that is orthogonal to U and lies in the plane spanned by U and V. N=(UxV)xU. Note that ax(bxc) =b(a.c)– c(a.b). For example, given a viewing direction U and an upward vertical direction V, the three vectors NxU, N, and U may be used to form the basis of a viewing coordinate system that would **avoid leaning the head** sideways. Furthermore, the surface normal N at a vertex A of a triangle mesh may be estimated as ABxAC+ACxAD+ADxAE+…+AHxAB, where B, C, D, E…H are the neighbors of A in the order in which they appear in a counter-clockwise walk around A.

The expression U•(VxW) is called a **mixed product**. It may also be computed as the **determinant** of the 3x3 matrix having vectors U, V, and W as columns. To illustrate the power of the mixed product, note that DA•(DBxDC)/6 is the signed **volume of a tetrahedron** with vertices A, B, C and D. If the triangle A, B, C appears clockwise from D, then DA•(DBxDC) is **positive**. The **volume** of a solid bounded by a **triangle mesh** may be computed as 1/6 of the sum over all triangles T of the mixed products DA•(DBxDC), where D is any fixed point, such as the origin of the coordinate system or the center of gravity of the vertices of the mesh and where A, B, and C are the vertices of T sorted counterclockwise with respect to its outward normal.

A **line** may be represented by a point P and a tangent direction T. We will refer to it as **line(P,T).**

A **plane** may be represented by a point Q and a normal direction N. We will refer to it as **plane(Q,N)**.

Note that all points on line(P,T) may be expressed in **parametric form** as L(s)=P+sT, where the scalar s is the parameter that defines the signed distance between point L(s) and P along T. All points A on plane(Q,N) satisfy the **implicit equation** AQ•N=0. Thus, the **line/plane intersection** I of plane(Q,N) with line(P,T) is defined by the value of s for which L(s)Q•N=0. Substituting P+sT for L(s) yields: (Q–P–sT)•N=0. Distributing the dot-product yields: PQ•N=sT•N. When T•N=0, the line is parallel to the plane. Otherwise, s=(QP•N)/(T•N).

Let I be the **tree-planes-intersection** of plane(A,U), plane(B,V), and plane(C,W). I satisfies AI•U=0, BI•V=0, and CI•W=0, which can be written **I**•U=–A•U, I•V=–B•V, and I•W=–C•W. These three equations form a linear system I^T(U V W)=–(A•U, B•V, C•W), which may be easily solved for the three coordinates of I if the determinant of the matrix (U V W) is not zero.

The signed **point/plane distance** between point P and plane(Q,N) is QP•N. The square of that distance is $(QP•N)^2$ and can be written as (P•N–Q•N)(N•P–Q•N), or $P^T(N^TN)P–2(Q•N)(N•P)+(Q•N)^2$. This is a second degree polynomial in the three coordinates of P and is represented by 10 coefficients. The whole expression may also be written in homogeneous coordinates using a 4x4 matrix M as H^TMH. Where H is a 4D vector constructed by appending a 1 as a fourth coordinate to P. Consider two planes and their quadrics, M_1 and M_2, as defined above. The sum of the squares of the distance of a point P to these two planes can be written as H^TMH, where M is M_1+M_2. Thus, the average of the squares of the distances between P and a set of planes may be easily computed as H^TMH, where M is the sum of the quadric matrices M_i for the planes. Setting to zero the derivatives with respect to the coordinates x, y and z of H^TMH yields three linear equations in x, y, and z. Solving them produces the point closest to all the planes in the least square sense.

Exercise 1: Use the operators introduced here to express the minimum distance between line(P,T) and line(Q,U).

Exercise 2: Formulate a test establishing whether point P lies inside tetrahedron with vertices A, B, C, D.